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Averaging and Reynolds Operators on Banach Algebras
II. Spectral Properties of Averaging Operators

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1. INTRODUCTION

This paper continues the discussion of averaging operators on Banach algebras which was begun by the second author in [1], as part of an examination of Reynolds operators and representation formulas for them. The latter class of operators is not discussed here (despite the title); however, some of their spectral properties are dealt with in a third paper in the series, by R. A. S. Fox and J. B. Miller.

We recall the notation of [1]. $\mathfrak{A}$ is a complex algebra, not necessarily commutative, with identity e, and $\mathfrak{B}(\mathfrak{A})$ is the Banach algebra of all bounded linear operators on $\mathfrak{A}$ into $\mathfrak{A}$. An averaging operator T is an element of $\mathfrak{B}(\mathfrak{A})$ satisfying simultaneously the two laws

$$\left.\begin{aligned} T(x \cdot Ty) &= Tx \cdot Ty \\ T(Tx \cdot y) &= Tx \cdot Ty \end{aligned}\right\} \quad (\text{all } x, y \in \mathfrak{A}). \tag{1.1}$$

The discussion of the spectral properties of an averaging operator is based on the simple device of introducing the element

$$t = Te;$$

it turns out that the spectral properties of T can, to a great extent, be expressed in terms of properties of t, and in this way the spectral analysis done in $\mathfrak{A}$ rather than in $\mathfrak{B}(\mathfrak{A})$. A similar phenomenon occurs to some degree with Baxter operators, see [2], and with Reynolds operators, see [3] and [4].

A degenerate form of averaging operator arises from the equations

$$T(xy) = Tx \cdot y, \quad T(xy) = x \cdot Ty \qquad (\text{all } x, y \in \mathfrak{A}). \tag{1.2}$$

Let us use the term *multiplier* for an operator $T \in \mathfrak{B}(\mathfrak{A})$ which satisfies (1.2), and is therefore clearly an averaging operator. (Wendel [5] uses the term "centralizer".)

More generally, we could consider operators with one-sided properties. Let us call an operator satisfying the first equation of (1.1) or of (1.2) a *left averaging* operator or *left multiplier* respectively, with similar definitions for right. Most of our results will be about two-sided operators, but it is worth remarking at one or two points on the relevance of a left or right property.

First note that if T is a left multiplier, then

$$Ty = ty \qquad (\text{all } y \in \mathfrak{A}),$$

that is, T is precisely left multiplication by Te; and all left multiplications are left multipliers. T is a multiplier if and only if t belongs to the center of $\mathfrak{A}$.

For any operator $K \in \mathfrak{B}(\mathfrak{A})$ define its *scalar subalgebra* in $\mathfrak{A}$ to be

$$\mathfrak{S}(K) = \{y : K(xy) = Kx \cdot y, \quad K(yx) = y \cdot Kx \qquad \text{for all} \qquad x \in \mathfrak{A}\}.$$

Then $\mathfrak{S}(K)$ is a closed subalgebra of A on which K reduces to a multiplier; and K is a multiplier if and only if $\mathfrak{S}(K) = \mathfrak{A}$. T is an averaging operator if and only if $\mathfrak{R}(T) \subseteq \mathfrak{S}(T)$; and then $\mathfrak{R}(T) = \mathfrak{S}(T)$ if and only if $e \in \mathfrak{R}(T)$. ($\mathfrak{R}(K)$ denotes the range of K.)

If T is a left averaging operator, then by definition its domain is $\mathfrak{A}$ and its range is clearly a subalgebra. We do not assume $Te = e$; however, we remark that this condition is equivalent to the two properties $e \in \mathfrak{R}(T)$, $T^2 = T$. Note that the restriction of T to its range is a left multiplier.

CONTENTS. We conclude this section by listing some notation. Then in Section 2 a formula is given for the resolvent of T, and it is shown that the spectra of T and t coincide except perhaps for 0, whose role is examined further. Section 3 is about the fine structure of the spectrum of T, which can be related to the status of $\lambda e - t$ in $\mathfrak{A}$. In Section 4, these results are applied to the algebra $C(X)$, using Kelley's representation theorem for averaging operators on $C_\infty(X)$. The paper ends with an example, in Section 5.

NOTATION. We write $\mathrm{Sp}(a)$ and $\mathrm{Res}(a)$ for the spectrum and resolvent set of a, respectively, for $a \in \mathfrak{A}$, and $R(\lambda, a) = (\lambda e - a)^{-1}$ for the resolvent. For $K \in \mathfrak{B}(\mathfrak{A})$ we write $\mathrm{PtSp}(K)$, $\mathrm{ConSp}(K)$ and $\mathrm{RdSp}(K)$ for the point, continuous, and residual spectrum, respectively.

It will be useful also to refer to the states of an operator, using the notation of Taylor [6], pp. 235–240, by which the states are specified as follows. (A bar denotes closure.)

$$(I) \quad \mathfrak{R}(K) = \mathfrak{A}$$

$$(II) \quad \overline{\mathfrak{R}(K)} = \mathfrak{A} \qquad \text{but} \qquad \mathfrak{R}(K) \neq \mathfrak{A}$$

$$(III) \quad \overline{\mathfrak{R}(K)} \neq \mathfrak{A}$$

(1) K is one-to-one and K^{-1} is bounded

(2) K is one-to-one and K^{-1} is unbounded

(3) K is not one-to-one.

We shall write $\lambda \in N$ or m or N_m to mean that $\lambda I - K$ is in state N or m or N_m, respectively. Thus

$$\operatorname{Res}(K) = \{\lambda : \lambda \in I_1\}, \qquad \operatorname{ConSp}(K) = \{\lambda : \lambda \in II_2\},$$
$$\operatorname{PtSp}(K) = \{\lambda : \lambda \in 3\}, \qquad \operatorname{RdSp}(K) = \{\lambda : \lambda \in III_1 \cup III_2\}.$$

When a is in the center of $\mathfrak{A}$ we write M_a for the operator of multiplication by a. We write "iff" to mean "if and only if."

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2. Resolvent Set of an Averaging Operator

Except for one or two remarks, we assume throughout this section that T is an averaging operator, satisfying both equations in (1.1).

Note first that $T^2x = T(eTx) = Te \cdot Tx = t \cdot Tx, = Tx \cdot t$ similarly. Also $T(tx) = T(Te \cdot x) = Te \cdot Tx = t \cdot Tx$, and likewise $T(xt) = Tx \cdot t$. Thus

$$T(tx) = t \cdot Tx = T^2x = Tx \cdot t = T(xt); \tag{2.1}$$

T coincides on its range with M_t, t commuting with all elements in the range. We shall use these properties frequently. Though t is in the center of $\mathfrak{R}(T)$, it need not be in the center of $\mathfrak{A}$.

Theorem 1. $\operatorname{Res}(t)\diagdown\{0\} \subseteq \operatorname{Res}(T)$; *and for* $\lambda \in \operatorname{Res}(t)\diagdown\{0\}$,

$$R(\lambda, T)\, x = \frac{1}{\lambda} x + \frac{1}{\lambda} R(\lambda, t) \cdot Tx \qquad (\textit{all } x \in A). \tag{2.2}$$

Proof. For $\lambda \in \operatorname{Res}(t)\diagdown\{0\}$ and arbitrary $x \in \mathfrak{A}$ write

$$Lx = \frac{1}{\lambda} x + \frac{1}{\lambda} R(\lambda, t) \cdot Tx,$$

defining thereby an operator $L \in \mathfrak{B}(\mathfrak{A})$. By (1.1),

$$(\lambda I - T) Lx = x + R(\lambda, t)\, Tx - \frac{1}{\lambda}\, Tx - \frac{1}{\lambda}\, T[R(\lambda, t)] \cdot Tx. \tag{2.3}$$

Now since $T[R(\lambda, t)(\lambda e - t)] = t$, we have

$$T[R(\lambda, t)] \cdot (\lambda e - t) = t.$$

Thus $T[R(\lambda, t)] = tR(\lambda, t)$, and when this is substituted in the last term in (2.3), the right-hand side reduces to x. Similarly one shows that $L(\lambda I - T)\, x = x$, using (2.1). Thus $(\lambda I - T)^{-1}$ exists, equals L, and the result follows. //

THEOREM 2. $\mathrm{Res}(T)\setminus\{0\} \subseteq \mathrm{Res}(t)$, *and*

$$R(\lambda, T)\, e = R(\lambda, t). \tag{2.4}$$

PROOF. Let $\lambda \in \mathrm{Res}(T)$, $\lambda \neq 0$. Write $R(\lambda, T)\, e = c$, so that $e = \lambda c - Tc$. Apply T; we get

$$\begin{aligned} t &= \lambda Tc - tTc \\ &= (\lambda e - t)\, Tc = (\lambda e - t)(\lambda c - e) \\ &= \lambda(\lambda e - t)\, c - \lambda e + t. \end{aligned}$$

Therefore $(\lambda e - t)\, c = e$. Similarly $c(\lambda e - t) = e$. Thus $\lambda \in \mathrm{Res}(t)$, and (2.4) holds. //

The same argument can be used if merely $\lambda I - T$ is inversible, so long as e is in the domain of $(\lambda I - T)^{-1}$. Thus [*] *If* $\lambda \notin \mathrm{PtSp}(T)$, $\lambda \neq 0$ and $e \in \mathfrak{R}(\lambda I - T)$, *then* $\lambda \in \mathrm{Res}(t)$ *and* $(\lambda I - T)^{-1} e = R(\lambda, t)$. Then Theorem 1 shows that $\lambda \in \mathrm{Res}(T)$, so that in fact $\mathfrak{R}(\lambda I - T) = \mathfrak{A}$.

We remark that in the proof of Theorem 1 we used only the left averaging property; the proof of Theorem 2 uses both left and right. The two theorems show that $\mathrm{Sp}(T)$ and $\mathrm{Sp}(t)$ coincide, except perhaps for the number 0. Accordingly, we now examine the role of 0.

THEOREM 3. *If* $0 \notin \mathrm{PtSp}(T)$, *or if* $\mathfrak{R}(T)$ *is dense in* $\mathfrak{A}$, *then* T *is a multiplier. In the first case,* t *is not a divisor of zero. In fact,* T *is multiplication by a central nondivisor of zero (namely* t*) if and only if* $0 \notin \mathrm{PtSp}(T)$.

PROOF. Suppose $0 \notin \mathrm{PtSp}(T)$, so that T is one-to-one. For arbitrary $x \in \mathfrak{A}$ write $y_1 = Tx - tx$, $y_2 = Tx - xt$. Then

$$Ty_1 = T^2x - T(Te \cdot x) = t \cdot Tx - Te \cdot Tx = 0,$$

so $y_1 = 0$. Similarly $y_2 = 0$, and so $T = M_t$. Since T is one-to-one, t cannot be a divisor of zero.

Suppose instead that $\mathfrak{R}(T)$ is dense in $\mathfrak{A}$. Then given $x \in \mathfrak{A}$ there is a sequence $\{x_n\}$ such that $x = \lim Tx_n$; and then, since T is bounded, $Tx = \lim tTx_n = tx$, and likewise $Tx = xt$. Thus again $T = M_t$. //

THEOREM 4. *If* $0 = \mathrm{Res}(T)$, *then* $0 \in \mathrm{Res}(t)$.

PROOF. Suppose $0 \in \mathrm{Res}(T)$. Then by Theorem 3, $T = M_t$. Since $\mathfrak{R}(T) = \mathfrak{A}$, there exists $u \in \mathfrak{A}$ such that $tu = e = ut$. Thus $0 \in \mathrm{Res}(t)$. Also $T^{-1} = M_{t^{-1}}$. //

A modification of this argument gives the following more general result: *If* $0 \notin \mathrm{PtSp}(T)$ *and* $e \in \mathfrak{R}(T)$, *then* $0 \in \mathrm{Res}(t)$, *and* $T^{-1}e = t^{-1}$. (This is the case $\lambda = 0$ of the previous remark [*].)

COROLLARY. $\mathrm{Res}(T) \subseteq \mathrm{Res}(t)$. *If* $0 \notin \mathrm{PtSp}(T)$ *then* $\mathrm{Res}(T) = \mathrm{Res}(t)$.

Proof of the second statement: if $0 \notin \mathrm{PtSp}(T)$ and $0 \in \mathrm{Res}(t)$ then by Theorem 3 T is multiplication by a regular central element and so $0 \in \mathrm{Res}(T)$.

If $0 \in \mathrm{Res}(t)$, it is still possible that $0 \notin \mathrm{Res}(T)$; but then by the corollary 0 must be $\mathrm{PtSp}(T)$. In fact, since $\mathrm{Res}(t)$ is open,

$$R(\lambda, t) = -(u + \lambda u^2 + \lambda^2 u^3 + \cdots)$$

for $|\lambda| <$ some δ, with $u = t^{-1}$; and then

$$R(\lambda, T) = \frac{1}{\lambda}(I - uT) - u^2 T - \lambda u^3 T - \cdots$$

by Theorem 1. If $uT \neq I$, then $R(\cdot, T)$ has a simple pole at 0. For an example, see the end of Section 5.

3. FINE STRUCTURE OF Sp(T)

Again, T is assumed to be an averaging operator throughout the section. We relate the fine structure of $\mathrm{Sp}(T)$ to the status of $\lambda e - t$ as a singular element in $\mathfrak{A}$. It is a straightforward matter to deduce the character of $\lambda e - t$ from the location of λ in $\mathrm{Sp}(T)$, but less so to go in the other direction.

We use the following standard terminology. An element $x \in \mathfrak{A}$ is a left divisor of zero if $xy = 0$ for some element $y \neq 0$, $y \in \mathfrak{A}$. Again, x is a left generalized divisor of zero if there exists a sequence $\{y_n\}$ in $\mathfrak{A}$ such that $\inf_n \| y_n \| > 0$ and $\lim_n xy_n = 0$. Similar definitions hold for right divisors and right generalized divisors of zero. An element is a divisor (generalized divisor) of zero if it is either a right or left divisor (generalized divisor) of zero. If it is both right and left, it is called two-sided. A divisor of zero is clearly a generalized divisor of zero.

The element y above will be called a codivisor of zero with x, the sequence $\{y_n\}$ a generalized codivisor of zero. Note that 0 is a divisor, but not a codivisor.

THEOREM 5. *For $\lambda \neq 0$, and T averaging,*

(i) *$\lambda \in \mathrm{PtSp}(T)$ implies that $\lambda e - t$ is a two-sided divisor of zero;*

(ii) *$(\lambda I - T)^{-1}$ exists but is unbounded implies that $\lambda e - t$ is a two-sided generalized divisor of zero;*

(iii) *$\lambda \in \mathrm{Res}(T)$ iff $\lambda e - t$ is a regular element.*

PROOF. (i) Let $\lambda \in \mathrm{PtSp}(T)$; then $(\lambda I - T)\, x = 0$ for some $x \neq 0$, and so $0 = T(\lambda I - T)\, x = (\lambda e - t)\, Tx$. But $Tx = \lambda x \neq 0$. Therefore $\lambda e - t$ is a left divisor of zero, and (by (2.1)) also a right divisor of zero.

(ii) Suppose that $(\lambda I - T)^{-1}$ exists but is unbounded. Then there is a sequence $\{y_n\}$ in $\mathfrak{A}$ such that $(\lambda I - T)\, y_n \to 0$ while $\liminf \| y_n \| > 0$. But then $Ty_n(\lambda e - t) = (\lambda e - t)\, Ty_n = T(\lambda I - T)\, y_n \to 0$, and $\liminf \| Ty_n \| > 0$ since $\lambda y_n - Ty_n \to 0$ and $\liminf \| y_n \| > 0$; thus $\lambda e - t$ is a two-sided generalized divisor of zero.

(iii) has been proved already in Section 2. //

We remark that if the statement of Theorem 5 is modified by requiring T to be left averaging and replacing "two-sided" by "left" in (i) and (ii), this modified statement is true.

Next, we show that $\mathrm{ConSp}(T)$ is always empty; in fact that $\mathfrak{R}(\lambda I - T)$ is never dense in A when $\lambda \in \mathrm{Sp}(T)$. The cases $\lambda \neq 0$, $\lambda = 0$ will be treated separately. First we have

LEMMA 1. *Let $\lambda \neq 0$. Then $\lambda \in \mathrm{Sp}(T)$ implies that $\mathfrak{R}(\lambda I - T)$ is not dense in $\mathfrak{A}$. Equivalently, $\lambda \in I_1 \cup III$ when $\lambda \neq 0$.*

PROOF. Suppose that $\mathfrak{R}(\lambda I - T)$ is dense in $\mathfrak{A}$, i.e., $\lambda \in I \cup II$. Then there exists a sequence $\{x_n\}$ in $\mathfrak{A}$ such that $e = \lim(\lambda I - T)\, x_n$, and so, since T is continuous,

$$t = \lim(\lambda e - t)\, Tx_n\,. \tag{3.1}$$

Suppose also that $(\lambda I - T)^{-1}$ exists but is unbounded, or that $(\lambda I - T)^{-1}$ does not exist, i.e., $\lambda \in 2 \cup 3$. Then $\lambda e - t$ is a two-sided generalized divisor of zero, by Theorem 5. Therefore there exists a sequence $\{y_m\}$ such that

$$\lim y_m(\lambda e - t) = 0, \qquad \liminf \| y_m \| > 0, \tag{3.2}$$

and it can be assumed without loss of generality that $\| y_m \| \leqslant M$ for some M, all m.

Given $\epsilon > 0$, there exists by (3.1) an integer n_0 such that

$$\| t - (\lambda e - t)\, Tx_{n_0} \| < \epsilon/2M,$$

and then for all m,

$$\| y_m t - y_m(\lambda e - t)\, Tx_{n_0} \| < \tfrac{1}{2}\epsilon.$$

But by (3.2) there exists an integer m_0 such that

$$\| y_m(\lambda e - t)\, Tx_{n_0} \| < \tfrac{1}{2}\epsilon \qquad \text{for} \quad m \geqslant m_0 .$$

Then $\| y_m t \| < \epsilon$ for $m \geqslant m_0$, so that $\lim y_m t = 0$. But then (3.2) is a contradiction.

Thus $\lambda \notin (I \cup II) \cap (2 \cup 3)$. So if $\mathfrak{R}(\lambda I - T)$ is dense in $\mathfrak{A}$, we must have $\lambda \in I_1 \cup II_1$, that is (since II_1 is impossible from the state diagram, [6] p. 237), $\lambda \in I_1$. The stated result follows. //

LEMMA 2. *If $0 \in \mathrm{Sp}(T)$, then $\mathfrak{R}(T)$ is not dense in $\mathfrak{A}$. That is, $0 \in I_1 \cup III$.*

PROOF. Suppose $0 \in \mathrm{Sp}(T)$ and $\mathfrak{R}(T)$ is dense in $\mathfrak{A}$. By Theorem 3, $T = M_t$. Now by the corollary to Theorem 4, $0 \in \mathrm{Res}(t)$ implies $0 \in \mathrm{Res}(T) \cup \mathrm{PtSp}(T)$. Therefore either t is singular, or $0 \in \mathrm{Res}(t) \cap \mathrm{PtSp}(T)$. If t is singular then $\mathfrak{R}(T)$, $= \{xt : x \in \mathfrak{A}\}$, consists of singular elements, and since the set of regular elements of $\mathfrak{A}$ is open and nonempty, $\mathfrak{R}(T)$ cannot be dense in $\mathfrak{A}$, contrary to assumption.

There remains the case where $\mathfrak{R}(T)$ is dense in $\mathfrak{A}$ and $0 \in \mathrm{Res}(t) \cap \mathrm{PtSp}(T)$. But this also is impossible, for since $T = M_t$, $0 \in \mathrm{PtSp}(T)$ implies $tx = 0$ for some $x \neq 0$, so that $0 \notin \mathrm{Res}(t)$. //

From these two lemmas follows:

THEOREM 6. *The continuous spectrum of an averaging operator is empty.*

To give a complete description of the location of λ in $\mathrm{Sp}(T)$ in terms of $\lambda e - t$, we introduce the further and terminology notation:

$$\begin{aligned} Z_\lambda &= \{x \in \mathfrak{A} : (\lambda e - t)\, x = 0 \quad \text{or} \quad x(\lambda e - t) = 0, x \neq 0\} \\ &= \text{set of codivisors of zero with } \lambda e - t, \\ Y_\lambda &= \text{set of all sequences } \{y_n\} \text{ from } \mathfrak{A} \text{ such that } \inf \| y_n \| > 0 \\ &\ \text{and} \quad \lim (\lambda e - t)\, y_n = 0 \quad \text{or} \quad \lim y_n(\lambda e - t) = 0 \\ &= \text{set of generalized codivisors of zero with } \lambda e - t, \\ \mathfrak{N}(T) &= \text{null space of } T. \end{aligned}$$

The first result concerns the converse of Theorem 5, (i):

LEMMA 3. *Let $\lambda \neq 0$. Then each of the following conditions* (a) *and* (b) *is separately a necessary and sufficient condition for $\lambda \in \mathrm{PtSp}(T)$:*

(a) $Z_\lambda \cap \mathfrak{R}(T) \neq \emptyset$; (b) $Z_\lambda \setminus \mathfrak{N}(T) \neq \emptyset$.

PROOF. Assume (a). Then there exists $x \in \mathfrak{A}$ such that $Tx \neq 0$ and $(\lambda e - t) \cdot Tx = 0$ or $Tx \cdot (\lambda e - t) = 0$. That is, supposing the first, there exists $y = Tx \neq 0$ such that $(\lambda I - T)\, y = 0$. Therefore $\lambda \in \mathrm{PtSp}(T)$.

Conversely suppose $\lambda \in \mathrm{PtSp}(T)$. Then there exists $x \neq 0$ such that $Tx = \lambda x$, and by application of T, $(\lambda e - t)\, Tx = 0$. Clearly $Tx \neq 0$, so $Tx \in Z_\lambda \cap \mathfrak{R}(T)$.

That (b) is necessary and sufficient follows from the fact that (a) and (b) are equivalent. In fact, $x \in \mathfrak{R}(T) \cap Z_\lambda$ implies $x \in Z_\lambda \setminus \mathfrak{N}(T)$, and $y \in Z_\lambda \setminus \mathfrak{N}(T)$ implies $Ty \in \mathfrak{R}(T) \cap Z_\lambda$. We omit the details. //

Corollary. *Let $0 \notin \mathrm{PtSp}(T)$. Then $\lambda \in \mathrm{PtSp}(T)$ if and only if $\lambda e - t$ is a two-sided divisor of zero.*

Proof. If $0 \notin \mathrm{PtSp}(T)$ then $\mathfrak{N}(T) = \{0\}$. Condition (b) becomes the stated property of $\lambda e - t$. //

The next result concerns the converse of Theorem 5, (ii).

Lemma 4. *Let $\lambda \neq 0$. Then each of the following conditions* (a′) *and* (b′) *is separately a necessary and sufficient condition for $(\lambda I - T)^{-1}$ to be unbounded or nonexistent, i.e., for $\lambda \in 2 \cup 3$.*

(a′) *There exists a sequence in Y_λ whose elements are in $\mathfrak{R}(T)$.*

(b′) *There exists a sequence $\{y_n\}$ in Y_λ for which $\{Ty_n\}$ is also in Y_λ.*

We omit the proof, which is similar to that of Lemma 3. There is also a corollary in terms of the spectral role of 0: *If $0 \in 1$, i.e., if T^{-1} exists and is bounded, then the converse of Theorem* 5, (ii) *holds.*

It has already been observed in Lemmas 1 and 2 that I_1 and III are the only possible states for $\lambda I - T$. Using the results above we can list necessary and sufficient conditions on t the four possible cases. This is done separately for $\lambda \neq 0$ and $\lambda = 0$ in the following theorems.

Theorem 7. *For $\lambda \neq 0$ and T averaging,*

(i) *$\lambda \in I_1$ iff $\lambda e - t$ is regular;*

(ii) *$\lambda \in III_1$ iff $\lambda e - t$ is singular but has no generalized codivisor of zero in $\mathfrak{R}(T)$;*

(iii) *$\lambda \in III_2$ iff $\lambda e - t$ has a generalized codivisor of zero in $\mathfrak{R}(T)$ but no codivisor of zero in $\mathfrak{R}(T)$;*

(iv) *$\lambda \in III_3$ iff $\lambda e - t$ has a codivisor of zero in $\mathfrak{R}(T)$.*

Proof. (i) This has been proved already in Section 2. (ii) By Lemma 1, $\lambda \in III_1$ iff $\lambda \in \mathrm{Sp}(Y) \cap 1$, and so by Theorems 1 and 2, iff $\lambda \in \mathrm{Sp}(t) \setminus (2 \cup 3)$. The result follows from Lemma 4, (a′). (iii) This follows immediately from Lemma 3, (a) and Lemma 4, (a′). (iv) This is Lemma 3, (a).

THEOREM 8. *For T averaging,*

(i) $0 \in I_1$ *iff* $T = M_t$ *and* t *is regular*;

(ii) $0 \in III_1$ *iff* $T = M_t$ *and* t *is singular but not a generalized divisor of zero*;

(iii) $0 \in III_2$ *iff* $T = M_t$ *and* t *is a generalized divisor of zero but not a divisor of zero*;

(iv) $0 \in III_3$ *iff either* $T = M_t$ *and* t *is a divisor of zero, or* $T \neq M_t$.

The proof uses Theorem 3 and 4 and Lemma 2, and is otherwise a straightforward consideration of the possible cases; we omit the details.

4. THE CASE $\mathfrak{A} = C(X)$

In this section we consider the case where $\mathfrak{A}$ is the Banach algebra $C(X)$ of all complex-valued continuous functions on a compact Hausdorff space X, with the supremum norm. For algebras of continuous functions there is the well-known representation theorem of Garrett Birkhoff and J. L. Kelley, which asserts that an averaging operator determines a partition of the ground space such that every function maps to a function constant on each set of the partition, with constant value an averaging of the values of the function on that set. The most succinct form of the theorem is Kelley's [7], which for the present purposes needs modification in two respects, since as stated by Kelley it applies to $C_\infty(Y, R)$, the algebra of real-valued functions vanishing at a selected point ∞ of a compact Hausdorff space Y. This algebra has no identity, so that the element t is not defined; moreover, spectral properties are more easily discussed in complex algebras.

To deduce the corresponding representation for $C(X, R)$, we need only identity functions in $C(X)$ with the corresponding functions in $C_\infty(X \cup \{\infty\}, R)$, where ∞ is an adjoined isolated point.

The extension of the theorem to complex-valued functions is also straightforward (as suggested in [7], p. 214): it is sufficient to make the natural adaption of Lemmas 1.3 and 1.4 of [7] and to use the more general form of the Riesz-Kakutani theorem (see [8], Theorem 4.10.1).

With these modifications, Kelley's theorem gives the following representation. Let T be any bounded linear operator on $C(X)$; for given $x \in X$, $f \to (Tf)(x)$ is a bounded linear functional on $C(X)$ and so there is determined uniquely a regular complex Borel measure μ_x on X such that

$$(Tf)(x) = \int f(y)\, \mu_x(dy) \qquad (\text{all } f \in C(X)). \tag{4.1}$$

Write

$$D_x = \{y \in X : (Tf)(y) = (Tf)(x) \quad \text{for all} \quad f \in C(X)\}.$$

The sets D_x are closed, and therefore measurable, and as x ranges over X determine a partition of X. Also, $\mu_y = \mu_x$ when $y \in D_x$. The averaging property for T is equivalent to the following: For every x the carrier of μ_x lies in D_x. (The carrier of μ is the set of all points z such that each neighbourhood of z contains a set E with $\mu(E) \neq 0$.) We propose to apply to this representation the results of Sections 2 and 3.

It is more convenient to have an independent index set for the distinct D's and μ's: let it be A, and write in particular

$$D_0 = \{y \in X : (Tf)(y) = 0 \quad \text{for all} \quad f \in C(X)\}, \qquad \mu_0 = 0. \tag{4.2}$$

D_0 is seen to be a set of the partition, and μ_0 the corresponding measure. Equation (4.1) becomes

$$(Tf)(x) = \int_{D_\alpha} f\, d\mu_\alpha \qquad \text{for} \quad x \in D_\alpha \qquad (\text{all } \alpha \in A). \tag{4.3}$$

For each α write

$$\omega_\alpha = \mu_\alpha(D_\alpha) = \mu_\alpha(X), \qquad \omega_0 = \mu_0(D_0) = 0.$$

Since e is the function identically equal to 1, the element t of previous sections is the function

$$t(x) = \omega_\alpha \qquad \text{for} \quad x \in D_\alpha \qquad (\text{all } \alpha \in A).$$

The numbers ω_α may not all be distinct. For $\beta \in A$ write

$$S_\beta = \bigcup \{D_\alpha : \omega_\alpha = \omega_\beta\} = \{x \in X : t(x) = \omega_\beta\},$$

$$S_0 = \bigcup \{D_\alpha : \omega_\alpha = 0\} = \{x \in X : t(x) = 0\}.$$

Suppose B indexes the distinct sets S_β; we obtain a coarser partition $\{S_\beta : \beta \in B\}$ of X than the Kelley partition. Since $S_\beta = t^{-1}(\omega_\beta)$, the sets S are closed and so measurable. Note that $D_0 \subseteq S_0$. Write

$$\Omega = \{\omega_\beta : \beta \in B\}.$$

It is clear that $\mathrm{Sp}(t) = \Omega$; for if $\lambda \notin \Omega$ then $(\lambda e - t)(x)$ is never zero and $\lambda \in \mathrm{Res}(t)$, and conversely. The resolvent is constant on the S's; in fact

$$R(\lambda, t)(x) = \frac{1}{\lambda - \omega_\beta} \qquad \text{for} \quad x \in S_\beta \qquad (\text{all } \beta \in B). \tag{4.4}$$

By the results of Section 2, $\mathrm{Sp}(T)$ is Ω, together perhaps with 0. The fine structure of $\mathrm{Sp}(T)$ turns out to be describable in various ways in terms of properties both of the sets D and S in X and of the set Ω in C. We give some theorems relating various situations. There are essentially four kinds of properties under consideration: the location of ω_β in $\mathrm{Sp}(T)$; the status of $\omega_\beta e - t$ in $C(X)$; topological properties of S_β (and component D's); whether or not ω_β is isolated in Ω. First, we have

THEOREM 9. *Let $\omega_\beta \neq 0$. Then either $\omega_\beta \in \mathrm{PtSp}(T)$ or $(\omega_\beta I - T)^{-1}$ exists but is unbounded. That is, for $\beta \neq 0$, $\omega_\beta \in 2 \cup 3$.*

PROOF. For positive integral n write

$$U_n = \left\{x : | t(x) - \omega_\beta | < \frac{1}{n}\right\}.$$

Then U_n is an open set containing S_β.

Choose some D_α contained in S_β. By Urysohn's lemma there is a continuous function f_n on X with range $[0, 1]$ such that $f_n(D_\alpha) = \{1\}$, $f_n(X \setminus U_n) = \{0\}$. Then

$$(Tf_n)(x) = \omega_\beta \quad \text{if} \quad x \in D_\alpha, \qquad 0 \quad \text{if} \quad x \in X \setminus U_n.$$

Therefore

$$\|(\omega_\beta I - T)\, Tf_n \| = \|(\omega_\beta e - t)\, Tf_n \| \leqslant \frac{1}{n} \| Tf_n \| \leqslant \frac{1}{n} \| T \|.$$

Thus with $g_n = Tf_n$, $\lim(\omega_\beta I - T) g_n = 0$ and $\| g_n \| \geqslant \omega_\beta$. The result follows. //

The next result, which distinguishes between the two possibilities in Theorem 9, requires a preliminary measure-theoretic lemma.

LEMMA 5. *Let μ be a regular Borel measure on a compact Hausdorff space X, and let O be an open subset of X for which $\mu(O) \neq 0$. Then there is a continuous function f on X with range* $[0, 1]$, *such that f vanishes on $X \setminus O$ and $\int f\, d\mu \neq 0$.*

PROOF. Since μ is regular, there exists a compact set K contained in O such that $\mu(K) \neq 0$. Let $| \mu(K)| = \beta$. By the regularity of $| \mu |$ there exists an open set V such that $K \subset V \subseteq O$ and $| \mu |(V \setminus K) < \beta$. By Urysohn's lemma there is a continuous function f on X with range $[0, 1]$ taking the value 1 on K and vanishing outside V; and then

$$\left| \int_K f\, d\mu \right| = \beta, \qquad \left| \int_{V \setminus K} f\, d\mu \right| < \beta,$$

so that $\int f\, d\mu \neq 0$. //

THEOREM 10. *Let $\omega_\beta \neq 0$. The following statements are equivalent:*

(i) $\omega_\beta \in \mathrm{PtSp}(T)$.

(ii) $S_\beta{}^0$, *the interior of* S_β, *contains some* D_α.

(iii) $S_\beta{}^0$ *contains a point in the carrier of some* μ_α.

When they hold, the eigenspace of ω_β *is the subalgebra consisting of all functions with supports in* S_β *and constant on all sets* D.

PROOF. Suppose (i). Then there exists $f \in C(X)$, $f \neq 0$, such that $(\omega_\beta I - T)f = 0$. Since $f \in \mathfrak{N}(T)$, f is constant on each D, and by (2.1)

$$(Tf)(x) = t(x)f(x) = \omega_\alpha f(x) \qquad \text{for} \quad x \in D_\alpha, \qquad \text{each } \alpha.$$

But $(Tf)(x) = \omega_\beta f(x)$ for all x, so f vanishes outside S_β and therefore also on ∂S_β, the boundary of S_β. But f must take a constant nonzero value on some D_α, and then this D_α must lie in $S_\beta{}^0$. Thus (i) implies (ii).

Suppose (ii); since D_α contains the carrier of μ_α and $\alpha \neq 0$, (iii) follows.

Finally, suppose (iii). Then by Lemma 5 applied to $O = S_\beta{}^0$ and $\mu = \mu_\alpha$, there exists an f in $C(X)$ vanishing outside $S_\beta{}^0$ and such that $\int f\, d\mu_\alpha \neq 0$. Thus Tf is not the zero function, but has its support in S_β. Therefore $(\omega_\beta I - T)\, Tf = (\omega_\beta e - t)\, Tf = 0$, so $\omega_\beta \in \mathrm{PtSp}(T)$. Thus (iii) implies (i).

The last statement of the theorem is clear. //

The following theorem summarizes these results describing the fine structure of $\mathrm{Sp}(T)$ in terms of the topological properties of the D's and S's.

THEOREM 11. *Let $\lambda \neq 0$. Then $\lambda \in I_1 \cup III_2 \cup III_3$; and*

(i) $\lambda \in I_1$ *iff* $\lambda \notin \Omega$;

(ii) $\lambda \in III_2$ (*i.e.*, $\lambda \in \mathrm{RdSp}(T)$) *iff* $\lambda \in \Omega$, *say* $\lambda = \omega_\beta$, *and each* D *in* S_β *contains a point of* ∂S_β;

(iii) $\lambda \in III_3$ (*i.e.*, $\lambda \in \mathrm{PtSp}(T)$) *iff* $\lambda \in \Omega$, *say* $\lambda = \omega_\beta$, *and* $S_\beta{}^0$ *contains some set* D_α.

PROOF. Theorems 7 and 9 combine to show that I_1, III_2 and III_3 are the only possibilities. The stated equivalences then follow by combining Theorems 7 and 10 and the fact that $\mathrm{Sp}(T)\setminus\{0\} = \Omega\setminus\{0\}$. //

Next come some results concerning among other things the topological role of ω_β in Ω.

THEOREM 12. *For all β, ω_β is an isolated point of Ω if and only if S_β is clopen; and then ω_β is a pole of $R(\cdot, T)$ and so lies in* $\mathrm{PtSp}(T)$. *The pole is simple if $\beta \neq 0$, and at most double if $\beta = 0$.*

Here "clopen" means both open and closed. Since S_β is of necessity closed, the qualification is that it be also open.

PROOF. Since X is compact and Ω with the induced topology is Hausdorff, the continuous function t maps closed subsets to closed subsets. But $\{\omega_{\beta'}\} = t(S_{\beta'})$ for all $\beta' \in B$; from this it follows that $\{\omega_\beta\}$ is open, i.e., ω_β is isolated, if and only if S_β is open.

Suppose that ω_β is an isolated point of Ω, $\lambda \in \mathrm{Res}(T)$ and $\lambda \neq 0$, and write $d(\lambda)$ for the distance of λ from $\mathrm{Sp}(T)$. By substituting the form (4.4) for $R(\lambda, t)$ in (2.2) we get, for $f \in C(X)$,

$$(R(\lambda, T)f)(x) = \frac{1}{\lambda} f(x) + \frac{1}{\lambda(\lambda - \omega_\beta)} (Tf)(x) \qquad \text{for} \quad x \in S_\beta \quad (\text{all } \beta \in B).$$

A simple calculation leads to

$$\| R(\lambda, T)\| \leqslant \frac{1}{|\lambda|}\left(1 + \frac{1}{d(\lambda)} \| T \|\right).$$

Now let $\lambda \to \omega_\beta$. Eventually $d(\lambda) = |\lambda - \omega_\beta|$, and so $\|(\lambda - \omega_\beta)^2 R(\lambda, T)\| \to 0$ if $\beta \neq 0$. Thus ω_β is a simple pole. A similar argument shows that 0, if an isolated point of $\mathrm{Sp}(T)$, is at most a double pole. A pole of the resolvent is necessarily in the point spectrum; cf. [9], p. 194. //

On the other hand, the points of $\mathrm{PtSp}(T)$ need not be isolated: for a counter-example, see Section 5.

We remark that the theorem can be weakened in part: *If there exists a collection $\{D_\gamma : \gamma \in \Gamma \subsetneq A\}$ such that $\bigcup_{\gamma \in \Gamma} D_\gamma$ is contained in S_β and is a clopen subset of X, then $\omega_\beta \in \mathrm{PtSp}(T)$.* Proof: Consider the characteristic function f of $\bigcup D_\gamma$; if $\bigcup D_\gamma$ is clopen then $f \in C(X)$ and $Tf = \omega_\beta f$.

THEOREM 13. *For $\beta \neq 0$, $\omega_\beta e - t$ is a generalized divisor of zero. It is a divisor of zero if and only if $S_\beta{}^0$ is not empty. If $\omega_\beta e - t$ is not a divisor of zero, then ω_β is an accumulation point of Ω.*

PROOF. Suppose $\beta \neq 0$. By Theorem 9, $\omega_\beta \in 2 \cup 3$, and so Theorem 7 shows that $\omega_\beta e - t$ is a generalized divisor of zero; in fact it has a generalized codivisor of zero in $\mathfrak{R}(T)$.

Suppose $S_\beta{}^0$ is nonempty. Choose $x \in S_\beta{}^0$ and using Urysohn's lemma construct a function $f \in C(X)$ such that $f(x) = 1$, $f(X \setminus S_\beta{}^0) = \{0\}$. Then $(\omega_\beta e - t) f = 0$ and ω_β is a divisor of zero. Conversely suppose that $\omega_\beta e - t$ is a divisor of zero, and let f be a nonzero function in $C(X)$ such that $(\omega_\beta e - t) f = 0$. Then f vanishes outside S_β and therefore on ∂S_β. It follows that $S_\beta{}^0 \neq \emptyset$.

If $\omega_\beta e - t$ is not a divisor of zero, Theorem 5 implies $\lambda \notin \mathrm{PtSp}(T)$ and then Theorem 12 implies that ω_β is not an isolated point. //

Finally, we describe the role of 0 in $\mathrm{Sp}(T)$.

LEMMA 6. *T is a multiplier if and only if each D_α, $\alpha \neq 0$, is a singleton.*

PROOF. Let $T = M_t$, and let x, y be any two points in $X \setminus S_0$, so that $t(x) \neq 0$, $t(y) \neq 0$. X being compact, there exists $f \in C(X)$ with $f(x) = 0$, $f(y) \neq 0$, and then $Tf = tf$ separates x and y, so that x and y belong to distinct D's. Thus each D, excepting possibly D_0, is a singleton. The converse is obvious. //

The following theorem completes the classification in Theorem 11.

THEOREM 14. *The number 0 is always in $I_1 \cup III_2 \cup III_3$; and*

(i) *$0 \in I_1$ iff S_0 is empty and every D_α, $\alpha \neq 0$, is a singleton;*

(ii) *$0 \in III_2$, i.e., $0 \in \mathrm{RdSp}(T)$, iff $S_0{}^0 = \emptyset$, $S_0 \neq \emptyset$, and every D_α, $\alpha \neq 0$, is a singleton;*

(iii) *$0 \in III_3$, i.e., $0 \in \mathrm{PtSp}(T)$, iff either $S_0{}^0 \neq \emptyset$, or some set D_α, $\alpha \neq 0$, is not a singleton.*

PROOF. We use the criteria of Theorem 8, noting that $0 \in I_1 \cup III$. First, (i) follows immediately from Theorem 8(i), Lemma 6, and the fact that t is regular if and only if $S_0 = \emptyset$.

Now if $S_0 \neq \emptyset$, a simple application of Urysohn's lemma shows the existence for every $n > 0$ of a function $f_n \in C(X)$ such that $\| tf_n \| < 1/n$, $\|f_n\| = 1$. Thus $S_0 \neq \emptyset$ implies that t is a generalized divisor of zero. Therefore $0 \notin III_1$.

Again, Urysohn's lemma shows that t is a divisor of zero if and only if $S_0{}^0 \neq \emptyset$; (ii) follows. Then (iii) is the remaining possibility. //

We remark that the above proof establishes part of Theorem 13 for the case $\beta = 0$.

The range of T does not seem easy to characterize. Clearly it is contained in the set of functions constant on the D's and zero on D_0. If S_0 is clopen, it can be shown that $\mathfrak{R}(T)$ contains all functions constant on each D and zero on D_0; in particular, if S_0 is empty then $\mathfrak{R}(T)$ is precisely the set of functions constant on each D. If S_0 is not clopen, the situation appears to be more complicated.

5. AN EXAMPLE

Examples of the various situations in Section 4 can be constructed by taking simple sets X in R^2, and using appropriate partitions and multiples of Lebesgue measure. We consider one only. For notational simplicity we identify R^2 with the complex plane, writing $z = re^{i\theta}$.

Let X consist of the closed unit disc $|z| \leqslant 1$, together with the closed annulus $2 \leqslant |z| \leqslant 3$, and define T by

$$(Tf)(z) = \begin{cases} \int_0^{2\pi} f(e^{i\theta})\, d\theta & (|z| \leqslant 1), \\ (3-r)\int_0^{2\pi} f(re^{i\theta})\, d\theta & (2 \leqslant |z| \leqslant 3). \end{cases}$$

It is easily verified that T is an averaging operator on $C(X)$, and the sets D are:

$D_1 = \{z : |z| \leqslant 1\}$, $\mu_1 =$ Lebesgue measure on $|z| = 1$,
$D_r = \{z : |z| = r\}$, $\mu_r = (3-r) \times$ Lebesgue measure on $|z| = r$,
$D_0 = \{z : |z| = 3\}$.

The function t takes the value 2π on D_1, and $(3-r)\,2\pi$ on D_r; so we write $S_1 = D_1 \cup D_2$, $S_r = D_r$ for $2 < r < 3$, $S_0 = D_0$, and $\omega_1 = 2\pi$, $\omega_r = (3-r)\,2\pi$ for $2 < r < 3$. Furthermore $S_1{}^0 = D_1$ and $\omega_1 \in \mathrm{PtSp}(T)$, a case of Theorem 11, (iii); in fact $Tf = 2\pi f$ where f is the characteristic function of D_1. By Theorem 14, $0 \in \mathrm{PtSp}(T)$, while by Theorem 11, $\omega_r \in \mathrm{RdSp}(T)$ for $2 < r < 3$. Thus $\mathrm{Sp}(T)$ is the real interval $[0, 2\pi]$, the end points being in $\mathrm{PtSp}(T)$.

Thus the point spectrum of an averaging operator need not consist only of isolated points.

The restriction of T to $C(D_1)$ gives an example illustrating the situation $0 \in \mathrm{Res}(t) \setminus \mathrm{Res}(T)$ discussed at the end of Section 2.

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